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UNIQUENESS AND STABILITY OF RIGID-PLASTIC CYLINDERS UNDER INTERNAL PRESSURE AND AXIAL TENSION

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Abstract—Uniqueness of deformation and stability of the equilibrium configuration of a long closed-ended cylinder of rigid-plastic material, obeying the von Mises yield criterion, are examined under internal pressure and axial tensile load. Sufficient conditions are derived for uniqueness of the current state of the finitely deformed cylinder. By considering a material model of the Ramberg-Osgood type, it is shown that uniqueness is guaranteed up to a stage when either of the loads (or both) attains a maximum. For such a material model, "pressure-tension interaction curves" are obtained for some values of the wall-ratio and the strain-hardening index. Under internal pressure and *small* tension, however, the possibility of a bifurcation preceding a stability loss is shown to exist for certain cylinder geometry and material hardening properties.

INTRODUCTION

In a series of papers, Hill[1–3] presented a mathematical theory of uniqueness of deformation and stability of rigid-plastic solids. For uniqueness, a certain functional of the difference of two virtual velocity fields was required to be positive; for stability in the classical dynamic sense, the requirement was the positive-definiteness of the same functional for a single virtual velocity field (a virtual field is to be interpreted as one which satisfies kinematic boundary conditions and is constitutively admissible). The general conclusion was that for rigid-plastic solids uniqueness implies stability but not necessarily vice-versa. The earlier work of Swift[4] contained only the stability investigation of rigid-plastic solids; the instability was supposed to occur when one or more loads reached a peak value.

Only a few problems appear to have been solved for rigid-plastic solids when two or more types of load act simultaneously. The stability of a thin-tube under internal pressure and tension was investigated by Swift[4], Hillier[5] and Yamada and Aoki[6], each using a different formulation. All these investigations were performed in the light of "thin-wall approximation". Moreover, these authors either restricted their analyses to the examination of stability alone or solved the problem of uniqueness of deformation by considering uniform modes only. Recent studies of Miles[7] and Storåkers[8] take full account of the admissible velocity fields in the respective problems and show that possibilities exist for a bifurcation to occur preceding a stability loss. The present investigation is undertaken to investigate such possibilities in long closed-ended cylinders under internal pressure and axial tension. The analysis is performed using Hill's sufficient condition for uniqueness, which, for the sake of completeness, is discussed next.

UNIQUENESS CRITERION

Suppose at some instant t during a process of continuing deformation, a part S_p of the surface of the body is subject to a uniform fluid pressure $p(t)$, with a given pressure-rate p. In addition, suppose that the nominal traction-rates $\dot{T}_j$ and the velocity v_j are prescribed on parts S_T and the remainder S_v, respectively, of the surface. Then, following Hill's[9] general treatment of configuration-dependent loadings, a sufficient condition to guarantee uniqueness of the subsequent incremental deformation is that

$$\int_V \Delta\dot{s}_{ij}\Delta_{j,i}\,\mathrm{d}V - p\int_{S_p} n_i\Delta v_{j,i}\Delta v_j\,\mathrm{d}S_p > 0 \tag{1}$$

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where V is the current volume of the body, n_i is the unit outward normal to the surface, the prefix Δ denotes the difference of corresponding quantities in two solutions, $\dot{s}_{ij}$ is the material derivative of the nominal (Lagrangian) stress s_{ij}, measured with respect to a fixed Cartesian frame of reference x_i at the instant t, and a comma signifies differentiation with respect to x_i. The preceding condition can be simplied for incompressible, isotropic rigid-plastic solids having rate-constitutive relation of the form[10]:

$$\begin{aligned} h\epsilon_{ij} &= m_{ij}\left(m_{kl}\frac{\mathcal{D}\tau_{kl}}{\mathcal{D}t}\right) \quad &\text{if} \quad m_{kl}\frac{\mathcal{D}\tau_{kl}}{\mathcal{D}t} > 0 \\ &= 0 \quad &\text{if} \quad m_{kl}\frac{\mathcal{D}\tau_{kl}}{\mathcal{D}t} \leq 0 \end{aligned} \tag{2}$$

Here, ϵ_{ij} and $\mathcal{D}_{ij}/\mathcal{D}t$ are the tensors of plastic strain-rate and the Jaumann derivative of the Kirchhoff stress, respectively, h is a positive scalar measure of the current rate of work-hardening and m_{ij} are the components of the unit outward normal to the current yield surface in 6-dimensional stress space. The uniqueness criterion (1) can now be written as (see, e.g. [7])

$$\int_V (h\lambda_{ij}\lambda_{ij} - \sigma_{ij}w_{k,i}w_{j,k})\,\mathrm{d}V - p\int_{S_p} n_i w_{i,j} w_j \,\mathrm{d}S_p > 0 \tag{3}$$

where σ_{ij} are the components of the Cauchy stress and $w_i \equiv v_i$ are incompressible velocity fields vanishing on S_v and are associated with strain-rate $\lambda_{ij} \equiv \Delta\epsilon_{ij}$ that are either zero or parallel to m_{ij} (though not necessarily in the same direction as m_{ij}). The application of the uniqueness condition (3) requires the construction of the most general velocity field compatible with direction of strain-rate given by the stress-distribution at a generic instant during the process of deformation.

FORMULATION OF THE PROBLEM

Consider a closed-ended rigid-plastic cylinder initially of length l_0, inner radius a_0 and outer radius b_0. Suppose that by some arbitrary combination of internal pressure and axial tension, the cylinder is deformed to the current state in which the material is everywhere at the yield-point and obeys the von Mises yield condition; in the current state let it be assumed that the dimensions of the cylinder are l, a and b. Provided that the cylinder is long, the state of stress should be effectively independent of the axial coordinate z, except near the ends. Furthermore, transverse plane sections should remain plane. The combined radial and axial strain-rate must, therefore, be expressible by the following velocity components in the cylindrical polar coordinate system r, θ, z:

$$u = -A_0 r + \frac{C_0}{r}; \qquad v = 0; \qquad w = 2A_0 z \tag{4}$$

where A_0, C_0 are arbitrary constants. Then, according to the flow rule

$$\frac{\partial u/\partial r}{u/r} = \frac{-A_0 - C_0/r^2}{-A_0 + C_0/r^2} = \frac{\sigma'_{rr}}{\sigma'_{\theta\theta}}$$

Here, the prime denotes the deviatoric part of the stress. If this is substituted in the von Mises yield condition

$$\sigma'^2_{rr} + \sigma'_{rr}\sigma'_{\theta\theta} + \sigma'^2_{\theta\theta} = k^2(r)$$

there results

$$\sigma'_{rr} = \frac{(1+\beta r^2)}{(1+3\beta^2 r^4)^{1/2}}k(r); \qquad \sigma'_{\theta\theta} = -\frac{(1-\beta r^2)}{(1+3\beta^2 r^4)^{1/2}}k(r)$$

$$\sigma'_{zz} = -\frac{2\beta r^2}{(1+3\beta^2 r^4)^{1/2}}k(r)$$

where $\beta = A_0/C_0$. Since the stresses are independent of z and θ, two of the three equilibrium equations are automatically satisfied; the equation of equilibrium in the radial direction is

$$\frac{d\sigma_{rr}}{dr} - \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} = 0. \tag{5}$$

Since $\sigma'_{\theta\theta} - \sigma'_{rr} = \sigma_{\theta\theta} - \sigma_{rr}$, we have

$$\frac{d\sigma_{rr}}{dr} = \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} = \frac{2k(r)}{r(1+3\beta^2 r^4)^{1/2}}. \tag{5a}$$

Hence, using the boundary condition on $r = b$,

$$\sigma_{rr} = -\int_r^b \frac{2k(r)}{r(1+3\beta^2 r^4)^{1/2}} dr \tag{6}$$

and, therefore,

$$\sigma_{\theta\theta} = \frac{2k(r)}{(1+3\beta^2 r^4)^{1/2}} - \int_r^b \frac{2k(r)}{r(1+3\beta^2 r^4)^{1/2}} dr. \tag{7}$$

Using the boundary condition on $r = a$, we find

$$p = \int_a^b \frac{2k(r)}{r(1+3\beta^2 r^4)^{1/2}} dr \tag{8}$$

where p is the internal pressure. The distribution of axial stress is

$$\begin{aligned} \sigma_{zz} &= \sigma_{rr} - (2\sigma'_{rr} + \sigma'_{\theta\theta}) \\ &= \frac{(1+3\beta r^2)}{(1+3\beta^2 r^4)^{1/2}} k(r) - \int_r^b \frac{2k(r)}{r(1+3\beta^2 r^4)^{1/2}} dr. \end{aligned} \tag{9}$$

Knowing σ_{zz}, the total axial load $\bar{T}$ is found to be

$$\bar{T} = \int_a^b \sigma_{zz} 2\pi r \, dr = \pi a^2 p + 6\pi \int_a^b \frac{\beta r^3 k(r)}{(1+3\beta^2 r^4)^{1/2}} dr$$

and, therefore, the applied axial load (tension) is

$$T = \bar{T} - \pi a^2 p = 6\pi \int_a^b \frac{\beta r^3 k(r)}{(1+3\beta^2 r^4)^{1/2}} dr. \tag{10}$$

From (8) and (10), it is clear that both the axial load and the pressure are functions of A_0, C_0. If the load and the pressure are specified, the values of A_0 and C_0 can be determined; in other words, for some particular values of A_0 and C_0, there is a definite combination of the load and the pressure which must be applied to produce given relative rates of extension and expansion. Consider, for example, the case of a thin-walled cylinder. The yield stress k may be assumed to be constant throughout the thickness t which is very small compared to the mean radius R. Then, from (8) and (10), it follows that

$$\beta R^2 = \frac{1}{3} \frac{T/\pi R^2}{p}. \tag{11}$$

ADMISSIBLE VELOCITY FIELD

Based on the current stress distribution, the directions of the strain-rate $\lambda_{ij} \equiv \Delta\epsilon_{ij}$ can be determined. The components of strain-rates in r, θ, z-directions should be proportional to the deviatoric stresses in the corresponding directions, and in addition, in cylindrical polar coordinates, they should satisfy

$$\lambda_{rr} + \lambda_{\theta\theta} + \lambda_{zz} = 0; \qquad \lambda_{r\theta} = \lambda_{rz} = \lambda_{\theta z} = 0. \tag{12}$$

Expressed in terms of the velocity components u, v, w (now interpreted as the components in the r, θ, z-directions, respectively, of the difference of velocity fields in two modes), (12) is written as

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \tag{13}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = 0 \tag{14}$$

$$\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} = 0 \tag{15}$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0. \tag{16}$$

A method of obtaining the most general solution of partial differential eqns (13)–(16), subject to the requirement that strain-rates are proportional to stress-deviators, is shown in the appendix. This solution for $\beta \neq 0$ is

$$\begin{aligned} u &= C\left(\frac{1}{r} - \beta r\right) + B\left\{(\log r - 1) + \frac{\beta}{2}(r^2 + 2z^2)\right\}\cos(\theta + \alpha) \\ v &= -B\left\{\log r + \frac{\beta}{2}(2z^2 - r^2)\right\}\sin(\theta + \alpha) \\ w &= 2\beta C z - 2\beta B r z \cos(\theta + \alpha) \end{aligned} \tag{17}$$

where C, B and α are arbitrary constants and the significance of $\beta = A_0/C_0$ is mentioned in the previous section. Since no kinematic boundary conditions are specified, (17) is the complete solution for the admissible velocity field. The terms with the constant C represent the uniform mode, while the other terms represent a possible bifurcation from this mode.

APPLICATION OF UNIQUENESS CRITERION

Introducing the physical components of velocity and considering the prevailing stresses, the terms $h\lambda_{ij}\lambda_{ij}$, $\sigma_{ij}w_{k,i}w_{j,k}$ and $n_i w_{i,j} w_j$ in (3) can be transformed to give

$$h\lambda_{ij}\lambda_{ij} = h\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \tag{18}$$

$$\begin{aligned} \sigma_{ij}w_{i,k}w_{k,j} = {} & \sigma_{rr}\left[\left(\frac{\partial u}{\partial r}\right)^2 + \frac{\partial v}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) + \frac{\partial w}{\partial r}\frac{\partial u}{\partial z}\right] \\ & + \sigma_{\theta\theta}\left[\left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)\frac{\partial v}{\partial r} + \left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right)^2 + \frac{\partial w}{\partial r}\frac{\partial u}{\partial z}\right] \\ & + \sigma_{zz}\left[\frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial v}{\partial z} + \left(\frac{\partial w}{\partial z}\right)^2\right] \end{aligned} \tag{19}$$

$$n_i w_{i,k} w_k = \begin{cases} -\left[\dfrac{\partial u}{\partial r}u + \left(\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}\right)v + \dfrac{\partial u}{\partial z}w\right] & \text{at the cylindrical surface} \\ \pm\left[\dfrac{\partial w}{\partial r}u + \dfrac{1}{r}\dfrac{\partial w}{\partial \theta}v + \dfrac{\partial w}{\partial z}w\right] & \text{at cylinder ends.} \end{cases} \tag{20}$$

If the velocity field (17) is used in expressions (10)–(20) and each term in (3) is integrated along the z- and the θ-directions, noting that functions in θ have a period 2π, we get

$$\int_V h\lambda_{ij}\lambda_{ij}\,\mathrm{d}V = 2\pi l\int_a^b h\Big[B^2\Big(\frac{1}{r}+3\beta^2r^3\Big)+2C^2\Big(\frac{1}{r^3}+3\beta^2r\Big)\Big]\,\mathrm{d}r \tag{21}$$

$$\begin{aligned}\int_V \sigma_{ij}w_{i,k}w_{k,j}\,\mathrm{d}V = \pi l\int_a^b \Big[&\sigma_{rr}\Big\{4B^2\Big(\beta r-\frac{1}{3}\beta^2l^2r\Big)+2C^2\Big(\frac{1}{r^3}+\frac{2\beta}{r}+\beta^2r\Big)\Big\}\\ &+\sigma_{\theta\theta}\Big\{2C^2\Big(\frac{1}{r^3}-\frac{2\beta}{r}+\beta^2r\Big)-\frac{4}{3}\beta^2B^2l^2r\Big\}\\ &+\sigma_{zz}\{4B^2\beta^2(r^3-2rl^2)+8C^2\beta^2r\}\Big]\,\mathrm{d}r\end{aligned} \tag{22}$$

$$p\int_{S_p} n_iw_{i,k}w_k\,\mathrm{d}S_p = p\pi l\Big[B^2\Big(1-2\beta a^2+\frac{8}{3}\beta^2l^2a^2-2\beta^2a^4\Big)+2c^2\Big(\frac{1}{a^2}-3\beta^2a^2\Big)\Big]. \tag{23}$$

It may be mentioned here that, in evaluating the preceding surface integral,† the contribution from the cylinder-ends is taken in an approximate manner since the velocity field (17) is valid only at sections sufficiently away from the closed-ends of the cylinder. However, if it is assumed that the ends are closed in such a manner that the edges are permitted to warp freely but not the central portion, the velocity distribution at $r = a$ may be considered to hold for $0 \leq r \leq a$ at the ends.

From the analysis in Section 3, we know that

$$\sigma_{\theta\theta} = \sigma_{rr} + \frac{2k(r)}{(1+3\beta^2r^4)^{1/2}}$$

$$\sigma_{zz} = \sigma_{rr} + \frac{(1+3\beta r^2)}{(1+3\beta^2r^4)^{1/2}}k(r).$$

Hence, (22) can be modified to read

$$\begin{aligned}\int_V \sigma_{ij}w_{i,k}w_{k,j}\mathrm{d}V = \pi l\int_a^b \Big[&\sigma_{rr}\Big\{2B^2\Big(\beta r+2\beta^2r^3-\frac{16}{3}\beta^2l^2r\Big)+4C^2\Big(\frac{1}{r^3}+3\beta^2r\Big)\Big\}\\ &+\frac{2k(r)}{(1+3\beta^2r^4)^{1/2}}\Big\{2C^2\Big(\frac{1}{r^3}-\frac{2\beta}{r}+\beta^2r\Big)-\frac{4}{3}B^2\beta^2l^2r\Big\}\\ &+\frac{(1+3\beta r^2)}{(1+3\beta^2r^4)^{1/2}}k(r)\{4B^2\beta^2(r^3-2rl^2)+8C^2\beta^2r\}\Big]\,\mathrm{d}r.\end{aligned} \tag{22a}$$

With the use of (5a) and the boundary conditions on $r = a$, b, each term involving σ_{rr} in (22a) can be integrated by parts. The resulting expression is combined with (23) to give

$$\begin{aligned}\int_V \sigma_{ij}w_{i,k}w_{k,j}\,\mathrm{d}V + p\int_{S_p} n_iw_{i,k}w_k\,\mathrm{d}S_p &= \pi lp[B^2(1-\beta^2a^4)]\\ &+\pi l\int_a^b \frac{2k(r)}{(1+3\beta^2r^4)^{1/2}}\Big[B^2(\beta^2r^3-2\beta r)+4C^2\Big(\frac{1}{r^3}-\frac{\beta}{r}\Big)\Big]\,\mathrm{d}r\\ &+\pi l\int_a^b \frac{6\beta^2rk(r)}{(1+3\beta^2r^4)^{1/2}}[2B^2\beta^2(r^3-2rl^2)+4C^2\beta^2r]\,\mathrm{d}r.\end{aligned} \tag{24}$$

In (24), the pressure p can also be expressed in terms of the yield stress $k(r)$ by using the relation (8); the expression (24) then becomes

$$\begin{aligned}\int_V \sigma_{ij}w_{i,k}w_{k,j}\,\mathrm{d}V + p\int_{S_p} n_iw_{i,k}w_k\,\mathrm{d}S_p &= \pi l\int_a^b \frac{2k(r)}{r(1+3\beta^2r^4)^{1/2}}\\ &\times\Big[4C^2\Big\{\frac{1}{r^2}-\beta+3\beta^3r^4\Big\}+B^2\{1-2\beta r^2-4\beta^3l^2r^4+\beta^2(6\beta r^6+r^4-a^4)\}\Big]\,\mathrm{d}r.\end{aligned} \tag{25}$$

†This difficulty does not arise if the cylinder-ends are open. However, it is a separate problem for further investigation.

Finally, combining (21) with (25), the uniqueness criterion (3) can be written more simply in the following form:

$$\int_a^b \left\{\frac{h(r)}{r}\left[2B^2(1+3\beta^2r^4)+4C^2\left(\frac{1}{r^2}+3\beta^2r^2\right)\right]-\frac{2k(r)}{r(1+3\beta^2r^4)^{1/2}}\right.$$
$$\left.\times\left[4C^2\left\{\frac{1}{r^2}-\beta+3\beta^2r^6\right\}+B^2\{1-2\beta r^2-4\beta^3l^2r^4+\beta^2(6\beta r^6+r^4-a^4)\}\right]\right\}\mathrm{d}r>0. \qquad (26)$$

For (26) to hold for arbitrary C, B it is necessary and sufficient that

$$\int_a^b \left[\frac{h(r)}{r^3}(1+3\beta^2r^4)-\frac{2k(r)}{r^3(1+3\beta^2r^4)^{1/2}}(1-\beta r^2+3\beta^3r^6)\right]\mathrm{d}r>0 \qquad (27)$$

$$\int_a^b \left[\frac{h(r)}{r}(1+3\beta^2r^4)-\frac{k(r)}{r(1+3\beta^2r^4)^{1/2}}\left(1-2\beta r^2+\beta^2r^4\left\{1-\frac{a^4}{r^4}+6\beta r^2-4\beta l^2\right\}\right)\right]\mathrm{d}r>0. \qquad (28)$$

Inequalities (27) and (28) form a sufficient condition for the uniqueness of deformation of the cylinder under internal pressure and tension.

DISCUSSION

Let us first consider the implications of the preceding inequalities in the context of thin-walled cylinders. As can be readily seen, it is sufficient to study the condition (27) alone. The variations of h and k through the thickness of the cylinder can be neglected, and (27) is immediately integrated to give

$$\frac{h}{k}>\frac{2(1-\beta R^2+3\beta^3R^6)}{(1+3\beta^2R^4)^{3/2}} \qquad (29)$$

where R is the mean radius of the cylinder. Using the relation (11), we can rewrite (29) as

$$\frac{h}{k}>\frac{2(1-\alpha+3\alpha^3)}{(1+3\alpha^2)^{3/2}} \qquad (29a)$$

in which $\alpha = T/3p\pi R^2$. For a material having the constitutive law (2) and obeying the von Mises yield condition, it can be shown (see, e.g. [5]) that the hardening parameter h is expressible as

$$h=\frac{2}{3}\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\bar{e}}=\frac{2}{3}\frac{\bar{\sigma}}{\bar{z}}, \qquad \frac{1}{\bar{z}}=\frac{1}{\bar{\sigma}}\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\bar{e}} \qquad (30)$$

where $\bar{\sigma}$ and $\bar{e}$ are, respectively, the genralized stress and the generalized strain, defined by the relations

$$\bar{\sigma}=\left(\frac{3}{2}\sigma'_{ij}\sigma'_{ij}\right)^{1/2}, \qquad \bar{e}=\left(\frac{2}{3}e_{ij}e_{ij}\right)^{1/2} \qquad (31)$$

and $\bar{z}$ denotes the length of the subtangent to the generalized stress-strain curve. Then, we get

$$\frac{h}{k}=\frac{2}{\sqrt{3}}\frac{1}{\bar{z}} \qquad (32)$$

since $\sigma'_{ij}\sigma'_{ij}=2k^2$. Using (32), the result (29a) reduces to

$$\frac{1}{\bar{z}}>\frac{\sqrt{(3)}(1-\alpha+3\alpha^3)}{(1+3\alpha^2)^{3/2}} \qquad (33)$$

which is the same as obtained by Kumar and Ariaratnam [11] from an independent analysis for a thin-walled cylinder. In the extreme cases $\alpha = 0, \infty$, (33) yields the familiar results

$$\frac{1}{\bar{z}} > \sqrt{3} \qquad \text{for internal pressure only } (\alpha = 0)$$

$$\frac{1}{\bar{z}} > 1 \qquad \text{for simple tension only } (1/\alpha = 0).$$

Returning to the analysis of thick-walled cylinders, the two inequalities (27) and (28) should be satisfied simultaneously for the deformation to be unique. Since the terms with the constant C in the velocity field (17) represent the uniform mode, the inequality (27) is equivalent to the condition that the internal pressure and/or the tension should not reach a maximum. We wish to find out whether or not (28) is satisfied whenever (27) is.

When the cylinder is subjected to internal pressure alone, conditions (27) and (28) reduce to those obtained in [8]. On the other hand, for the case of simple tension we get

$$C^2 \int_a^b r\left\{h - \frac{2}{3}(k\sqrt{3})\right\} dr > 0 \tag{34}$$

and

$$B^2 \int_a^b r^3\left\{h - \frac{2(k\sqrt{3})}{3}\left[1 - \frac{2}{3}\frac{l^2}{r^2}\right]\right\} dr > 0. \tag{35}$$

The condition (35) is sufficient for uniqueness, because then (36) holds too. Therefore, uniqueness is guaranteed so long as the tension does not attain a peak value. This result was first given by Hill[2].

In general, the variations of h and k with r must be taken into account. For the von Mises solid, h can be expressed by the relation (30). Taking the logarithmic strain measure, we can write

$$e_{rr} = \log\frac{r}{r_0}, \qquad e_{zz} = \log\frac{l}{l_0}, \qquad e_{\theta\theta} = -(e_{rr} + e_{zz})$$

so that

$$\bar{e} = \frac{2}{\sqrt{3}}\left[\left(\log\frac{r}{r_0}\right)^2 + \left(\log\frac{l}{l_0}\right)^2 + \left(\log\frac{l}{l_0}\log\frac{r}{r_0}\right)\right]^{1/2}. \tag{36}$$

In (36), r is the current radial distance of a particle which was initially at the radius r_0, and l_0, l are the initial and the current lengths of the cylinder, respectively. The relation (36) is supplemented by the incompressibility condition

$$(r^2 - a^2)l = (r_0^2 - a_0^2)l_0$$

thereby making it possible to relate all the parameters in the current state to those in the reference state. The yield stress k is considered to be a function of the generalized strain and, based on the Ramberg–Osgood relation in classical plasticity, is assumed to be represented by the relation

$$k = k_0\bar{e}^m \tag{37}$$

where k_0 and m are material constants. Now, using (30), (31), (36) and (37), all the integrals in the inequalities (27)–(28) can be evaluated. Such a calculation was performed for cylinders having wall-ratios (b_0/a_0) up to 3.0 and for the range of strain-hardening index $0.05 \leq m \leq 0.50$. It was observed that the condition (28) was always satisfied at strains large enough that the condition (27) started to fail. This means that the deformation in this case is unique so long as the pressure or the tension does not reach a maximum. Nevertheless, the contrary may be true for some material models other than (37). For example, in the case of cylinders made-up of composite shells, the situation may be entirely different. In each case, the conditions (27)–(28) require a thorough investigation and, at this stage, no further conclusion can be drawn.

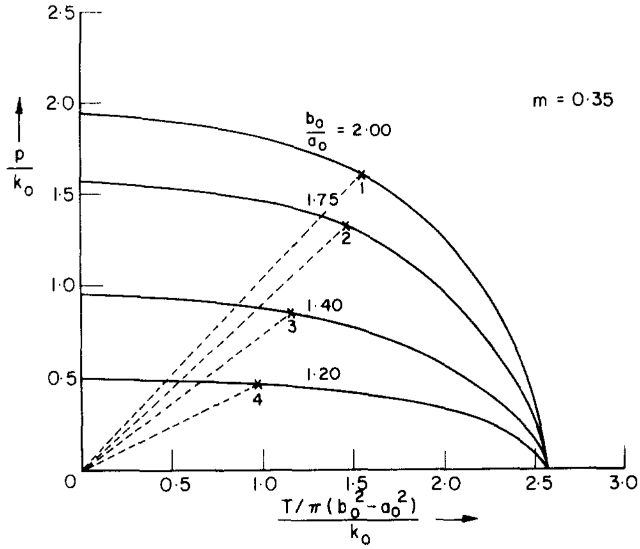

Fig. 1. Pressure tension interaction curves. $\overset{\times}{1}$ means $\frac{T/\pi(b_0^2 - a_0^2)}{p}$ equals 0.95; $\overset{\times}{2}$ means $\frac{T/\pi(b_0^2 - a_0^2)}{p}$ equals 1.09; 3 means $\frac{T/\pi(v_0^2 - a_0^2)}{p}$ equals 1.35; $\overset{\times}{4}$ means $\frac{T/\pi(b_0^2 - a_0^2)}{p}$ equals 2.08.

When the condition (27) is the governing one during the process of continuing deformation, either the pressure or the tension (or both) reaches the peak value. Using relations (8), (10), (36) and (37) the pressure and the axial load were calculated for increasing values of the expansion (a/a_0) and the extension (l/l_0). Three situations were found to occur: (i) Pressure reaching a maximum, while axial load is still increasing. (ii) Axial load reaching a maximum, while pressure is still increasing. (iii) Both pressure and axial load reaching a maximum simultaneously. 'Pressure-tension interaction curves' were drawn by taking the maximum value of the pressure (axial load) and the corresponding value of the axial load (pressure). Figure 1 shows such curves for particular value of m and the wall-ratios. In each case, the non-dimensional values of pressure p/k_0 and tension $T/k_0\pi(b_0^2 - a_0^2)$ are plotted. It may be noted that for every value of b_0/a_0 the ratio $T/p\pi(b_0^2 - a_0^2)$ has a definite value which corresponds to the case (iii) above. For b_0/a_0 equal to 1.20, 1.40, 1.75, 2.00, these values are found to be 2.08, 1.35, 1.09, 0.95, respectively. Value less than these correspond to the case (i) while the greater ones correspond to the case (ii).

As is well known, and also seen from the present analysis, in the case of simple tension no bifurcation is possible before the attainment of a maximum load. However, for a cylinder under internal pressure alone, the possibility exists of a bifurcation before the loss of stability [8]. It is worth investigating, therefore, what happens when the cylinder is subjected to internal pressure and a small tension. Conditions governing the uniqueness of deformation in such a case can be obtained from (27)–(28) by setting the terms in β^2 equal to zero; they become

$$\int_a^b \left[\frac{h}{r^3} - \frac{2k(1-\beta r^2)}{r^3}\right] dr > 0 \tag{27a}$$

and

$$\int_a^b \left[\frac{h}{r} - \frac{k(1-2\beta r^2)}{r}\right] dr > 0. \tag{28a}$$

For the most general case, not assuming material homogeneity, all that may be established is that if (27a) holds for $h > 0$ in $a \leq r \leq b$, then

$$\int_a^b \left[\frac{h}{r} - \frac{k(1-2\beta r^2)}{r}\right] dr > \int_a^b \left[\frac{h}{r} - \frac{k(1-2\beta a^2)}{r}\right] dr$$
$$> \int_a^b \left[\frac{h}{r} - \frac{b^2 k(1-2\beta a^2)}{r^3}\right] dr$$

$$> \int_a^b \left[\frac{h}{r} - \frac{hb^2(1-2\beta a^2)}{2r^3(1-\beta b^2)}\right] dr$$

$$> \left[1 - \frac{b^2(1-2\beta a^2)}{2a^2(1-\beta b^2)}\right] \int_a^b \frac{h}{r} dr$$

and consequently (28a) holds if

$$\frac{b}{a} \leqslant \sqrt{\left(\frac{2(1-\beta b^2)}{(1-2\beta a^2)}\right)}. \tag{38}$$

Otherwise, non-uniqueness may follow and there exists a possibility for the occurrence of a bifurcation preceding a stability loss. In this event, it means that the cylinder can still withstand an increase in pressure and tension.

CONCLUDING REMARKS

It has been shown that in the case of thick-walled cylinder subjected to an arbitrary combination of internal pressure and tension, the uniqueness is normally guaranteed up to the loss of stability (i.e. when either of the two or both loads reach a maximum). This is true at least for the material model described by the relation (37) which is similar to the Ramberg–Osgood relation in classical plasticity. For composite shells, however, the situation may be quite different. It has been found that, for a cylinder under pressure and *small* tension, and without any assumption of material homogeneity, the possibility exists of a bifurcation before stability loss.

"Pressure-tension interaction curves" have been obtained for some values of the wall ratio and the strain-hardening index. For each wall-ratio, there is a point on the curve which gives maximum values of the pressure and the tension, attained simultaneously during the process of deformation. This point separates each curve in two regions; one region corresponds to the occurrence of the maximum pressure first while the other one to the maximum tension first.

It may be mentioned here that Hill's uniqueness criterion employed in this investigation is one of sufficiency only. Hence, the critical stresses are merely lower bounds with respect to the loss of uniqueness. However, it is known that, for at least some of the problems studied so far in the rigid-plastic theory using Hill's method, the uniqueness conditions have proved to be also necessary. Hence, it may be expected that the conditions, derived in this paper for the uniqueness of deformation, are necessary as well. A definite conclusion can be drawn only by solving the related rate-problem, i.e. from the complete bifurcation solution.

Although the constitutive relation (2) is sufficiently general, the formulations presented in this investigation are for a material obeying the von Mises yield criterion, isotropic hardening rule and small strains. Moreover, the deformation theory of plasticity has been adopted for numerical calculations, as suggested by relation (36) where in the concept of generalized strain is used. From eqns (6) and (7) it is obvious though that in general proportional stressing does not prevail during the entire deformation course; use of the deformation theory is, therefore, an approximation. Strictly, an incremental theory should be adopted. It is expected, however, that for cylinders with small wall-thickness and for not very large plastic deformations, the results obtained under present approximation will be close enough to those resulting from an incremental approach.

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APPENDIX

Equations (13)–(16) can be rewritten as

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}=0 \tag{39}$$

$$\frac{1}{r^2}\frac{\partial u}{\partial \theta}+\frac{\partial}{\partial r}\left(\frac{v}{r}\right)=0 \tag{40}$$

$$\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial \theta}=0 \tag{41}$$

$$\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}=0. \tag{42}$$

For solving the above set of partial differential equations, first consider eqns (40)–(42). Equation (40) can be satisfied by choosing an arbitrary function q such that

$$u=r^2\frac{\partial q}{\partial r},\quad v=-r\frac{\partial q}{\theta}. \tag{43a}$$

Similarly, two arbitrary functions f and g may be chosen in order to satisfy equations (42) and (41), respectively:

$$u=\frac{\partial f}{\partial r},\quad w=-\frac{\partial f}{\partial z} \tag{43b}$$

$$v=\frac{1}{r}\frac{\partial g}{\partial \theta},\quad w=-\frac{\partial g}{\partial z}. \tag{43c}$$

From (43a)–(43c), we immediately obtain:

$$\frac{\partial f}{\partial r}-r^2\frac{\partial q}{\partial r}=0 \tag{44}$$

$$\frac{1}{r}\frac{\partial g}{\partial \theta}+r\frac{\partial q}{\partial \theta}=0 \tag{45}$$

$$\frac{\partial f}{\partial z}-\frac{\partial g}{\partial z}=0. \tag{46}$$

Equations (45) and (46) can be integrated to give

$$g+r^2q=H(r,z) \tag{47}$$

$$f-g=F(r,\theta) \tag{48}$$

where H, F are arbitrary functions. From (44), (47) and (48), the functions f, g, q are found to be

$$g=H(r,z)-r\int^r\frac{1}{2r'}\frac{\partial}{\partial r'}[H(r',z)+F(r',\theta)]\,\mathrm{d}r'-rG(\theta,z)$$

$$f=F(r,\theta)+H(r,z)-r\int^r\frac{1}{2r'}\frac{\partial}{\partial r'}[H(r',z)+F(r',\theta)]\,\mathrm{d}r'-rG(\theta,z)$$

$$q=\frac{1}{r}\int^r\frac{1}{2r'}\frac{\partial}{\partial r'}[H(r',z)+F(r',\theta)]\,\mathrm{d}r'+\frac{1}{r}G(\theta,z)$$

where function $G(\theta,z)$ is arbitrary. Hence, using (43a)–(43c), we get

$$u=\frac{1}{2}\frac{\partial}{\partial r}[F(r,\theta)+H(r,z)]-\int^r\frac{1}{2r'}\frac{\partial}{\partial r'}[H(r',z)+F(r',\theta)]\,\mathrm{d}r'-G(\theta,z) \tag{49}$$

$$v=-\int^r\frac{1}{2r'}\frac{\partial^2}{\partial r'\partial\theta}F(r',\theta)\,\mathrm{d}r'-\frac{\partial}{\partial\theta}G(\theta,r) \tag{50}$$

$$w=r\int^r\frac{1}{2r'}\frac{\partial^2}{\partial r'\partial z}H(r',z)\,\mathrm{d}r'-\frac{\partial}{\partial z}H(r,z)+r\frac{\partial}{\partial z}G(\theta,z). \tag{51}$$

If, for convenience, we define two functions $Q(r,\theta)$ and $P(r,z)$ in the form

$$Q(r,\theta)=\int^r\frac{1}{2r'}\frac{\partial}{\partial r'}F(r',\theta)\,\mathrm{d}r'$$

$$P(r,z)=H(r,z)-r\int^r\frac{1}{2r'}\frac{\partial}{\partial r'}H(r',z)\,\mathrm{d}r'$$

eqns (49)–(51) are simplified and become

$$u=r\frac{\partial Q(r,\theta)}{\partial r}+\frac{\partial P(r,z)}{\partial r}-Q(r,\theta)-G(\theta,z) \tag{52}$$

$$v = -\frac{\partial Q(r,\theta)}{\partial \theta} - \frac{\partial G(\theta,z)}{\partial \theta} \tag{53}$$

$$w = -\frac{\partial P(r,z)}{\partial z} + r\frac{\partial G(\theta,z)}{\partial z}. \tag{54}$$

Expressions (52)–(54) constitute the general solution of the eqns (40)–(42). The remaining eqn (39) can be used to solve for $P(r, z)$, $Q(r, \theta)$, $G(\theta, z)$, remembering that functions Q and G are periodic in θ. The general solution of the set of partial differential eqns (39)–(42) then becomes (excluding all rigid translations and rotations):

$$\begin{aligned} u &= \frac{C + C_1 z}{r} + B(\log r - 1)\cos(\theta + \alpha) - D(r^2 + 2z^2)\cos(\theta + \alpha) \\ &\quad - \sum_{n=2}^{\infty} \frac{p_n}{n}\left[\cos\left\{(n^2-1)^{1/2}\log\frac{r}{r_n}\right\} + (n^2-1)^{1/2}\sin\left\{(n^2-1)^{1/2}\log\frac{r}{r_n}\right\}\right]\cos(n\theta + \alpha_n) \\ &\quad - \int_{\nu} \frac{1}{\nu^2} J_1(\nu r)[E_\nu \cos \nu z + F_\nu \sin \nu z]\, d\nu \\ v &= -B\sin(\theta + \alpha)\log r + D(-r^2 + 2z^2)\sin(\theta + \alpha) \\ &\quad + \sum_{n=2}^{\infty} p_n \cos\left\{(n^2-1)^{1/2}\log\frac{r}{r_n}\right\}\sin(n\theta + \alpha_n) \\ w &= -C_1 \log\frac{r}{r_0} + 4Drz\cos(\theta + \alpha) - \int_{\nu} \frac{1}{\nu^2} J_0(\nu r)[-E_\nu \sin \nu z + F_\nu \cos \nu z]\, d\nu. \end{aligned} \tag{55}$$

where C, C_1, B, D, p_n, n, r_n, E, F are arbitrary constants and J_0, J_1 are the Bessels functions of the first kind of the order zero and one, respectively. The solution (55) is then made to satisfy the requirement that the strain rates in the r, θ, z-directions are proportional to the deviatoric stresses (see sec. 3) in the corresponding directions; for example

$$\frac{\partial u}{\partial r}\Big/\left(\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}\right) = -(1 + \beta r^2)/(1 - \beta r^2). \tag{56}$$

This requirement precludes the presence of certain terms in the general solution (55). The only admissible solution for $\beta \neq 0$ is given by (17).

For the case $\beta \to \infty$, i.e. when cylinder is subjected to axial tension alone, the velocity field given in [2] should be used since it has four arbitrary constants in contrast to two constants in the solution obtained from (17). However, it is known that the inequalities associated with these two extra constants, are not the critical ones for uniqueness. When $\beta = 0$, i.e. the cylinder is subjected to pressure alone, the actual solution is wider than that obtained from (17), as given in [8]. This is not a matter of surprise. The situation, in which a wider class of velocity field is available for plane strain (essentially corresponding here to the case $\beta = 0$) than for the genuinely three-dimensional case, is a familiar one when dealing with rigid/plastic materials with a smooth yield surface[2].